REFERENCE SOLUTION OF PROBLEM 37

Problem 37. *Let κ be an infinite regular cardinal. Suppose there is a sequence $(A_\alpha \mid \alpha < \kappa)$ of pairwise almost disjoint subsets of κ (i.e. for $\alpha \neq \beta < \kappa$: $|A_\alpha \cap A_\beta| < \kappa$). Show that there is a set $A \subseteq \kappa$, $|A| = \kappa$ that is almost disjoint from all A_α, $\alpha < \kappa$.*

Proof.

Claim. *For all $\alpha < \kappa$, $\left|\kappa \setminus \bigcup_{\beta<\alpha} A_\beta\right| = \kappa$.*

Proof. Assume this is false for some $\alpha < \kappa$. Note that $\kappa = (\bigcup_{\beta<\alpha} A_\beta) \cup (\kappa \setminus \bigcup_{\beta<\alpha} A_\beta)$. Since $|A_{\alpha+1}| = \kappa$, and by assumption $\left|\kappa \setminus \bigcup_{\beta<\alpha} A_\beta\right| < \kappa$, $\left|A_{\alpha+1} \cap (\bigcup_{\beta<\alpha} A_\beta)\right| = \kappa$.

But this is false: $A_{\alpha+1} \cap (\bigcup_{\beta<\alpha} A_\beta) = \bigcup_{\beta<\alpha}(A_{\alpha+1} \cap A_\beta)$. But this is a union of size less than κ of sets of size less than κ (since $A_{\alpha+1}$ is almost disjoint from A_β, $\beta < \alpha$). Because κ is regular, this union can not have size κ. Thus we have reached a contradiction □

Now construct $A = \{a_\alpha \mid \alpha < \kappa\}$ recursively. Take some $\alpha < \kappa$ and suppose a_β is known for all $\beta < \alpha$. Notice that $(\kappa \setminus \bigcup_{\beta<\alpha} A_\beta) \setminus \{a_\beta \mid \beta < \alpha\} \neq \emptyset$ by the claim. Then take $a_\alpha \in (\kappa \setminus \bigcup_{\beta<\alpha} A_\beta) \setminus \{a_\beta \mid \beta < \alpha\}$ arbitrarily.

Clearly, by construction, the a_α are pairwise distinct, so A has cardinality κ. Let $\alpha < \kappa$ and consider $A_\alpha \cap A$. By construction, for all $\beta > \alpha$, $a_\beta \notin A_\alpha$. Hence $|A_\alpha \cap A| \leq \alpha + 1 < \kappa$, i.e. A is almost disjoint to any A_α. □